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ABSTRACT

This paper is a report of a symposium on research in mathematical learning and teaching held during the annual meeting of the California Mathematics Council in 1966. Speakers and their topics were: Professor Frederick J. McDonald - "The Teaching of Mathematics"; Dr. John E. Coulson - "The Learning of Mathematics"; Professor Zoltan P. Dienes - "Research and Evaluation in Mathematics Learning". Introductory remarks were made by Dr. Jerry P. Becker. (Author/FL)

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Research in the Teaching and Learning of Mathematics*

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* This paper is a report of a symposium on research in mathematics learning and teaching held at Asilomar, California, during the annual meeting of the California Mathematics Council (Northern Section) December 2–4, 1966. The symposium was co-sponsored by the Stanford University chapter of Phi Delta Kappa.

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It is a pleasure for me to introduce the members of the panel. Professor Frederick J. McDonald will be speaking on "The Teaching of Mathematics"; Dr. John E. Coulson will speak on "The Learning of Mathematics" and Professor Zoltan P. Dienes will be speaking on "Research and Evaluation in Mathematics Learning". We are very sorry that Professor John Kelly is unable to be here, he was called away at the last minute.

I would like to mention a couple of things in an attempt to get us into the theme of this symposium, namely, "Research in the Teaching and Learning of Mathematics".

First, it seems appropriate that Phi Delta Kappa is co-sponsoring this symposium, for Phi Delta Kappa has long had research as one of its major cornerstones. Similarly, I think it is appropriate for me to highlight the fact that the inclusion of a research symposium such as this on the program of this Conference of the California

Mathematics Council indicates that this organization is also vitally interested in *research* as a means of discovering more about how children learn mathematics and how mathematics can most effectively be taught.

Moreover, it seems entirely appropriate that we have represented in the panel here both mathematicians and psychologists. If you think, for a moment, about the acts of *Learning* mathematics and of *teaching* mathematics, I think you will agree that both are mathematical *and* psychological in nature: mathematical in that the *content* being conveyed is mathematical, and psychological in that the *process* is a psychological process.

In general, I think there exist many mutual concerns of mathematics educators and psychologists. For example, a review of recent educational and psychological literature points to the fact that psychologists are now engaging in research which has relevance to mathematics education. Moreover, many general problems of learning seem to take on their sharpest and clearest form in a mathematical context. It would appear then that psychological research which involves mathematics can be of significance in terms of improving the processes of teaching and learning mathematics for the mathematics educators, and in terms of improving understanding of human behavior for the psychologists.

Well, I do not say that all people will agree with what I have said, but I hope this gets us into our topic for the morning. My fellow panel members may or may not touch upon what I have said, but, at any rate, I will turn it over to them at this time.

Professor Frederick J. McDonald, Stanford University

My purpose here today is to describe what can be done in the training of teachers, either beginning teachers or experienced teachers, to make them effective teachers of mathematics. I, and my colleagues, have been working on experimentation in teacher training for several years and very intensively in the last two years. We have developed a technology of training applicable to the training of teachers of many different subject areas. It is now sufficiently developed that it can be used to train teachers of particular subjects, such as teachers of mathematics, and can also be adapted to studying

the problems of teaching in a particular area. It is these potentialities that I would like to describe to you. I also want to inform you about what has already been accomplished.

When one talks about teaching mathematics it seems to me he ought to distinguish among three things that he might want to do as a teacher of mathematics. One of these, the most obvious one, is to teach children the basic processes and content of formal mathematics —number facts, various mathematical processes, the whole range of skills and understanding that usually are included in the subject area called mathematics. A second thing you may want to do is teach people to think quantitatively. It is not at all clear that if that were your primary goal, that the only way it could be achieved is through formal course work in mathematics. However, certainly, to teach people to think quantitatively, one would expect mathematicians and mathematics teachers to be integrally involved in whatever educational system was developed for this progress. A third reason for teaching mathematics might be to develop creative mathematicians; people who innovate in mathematics and who would solve some of the problems of formal mathematics.

Now, if you pick one, or all three of these goals, I think it is obvious that what you do about the teaching of mathematics probably shifts or changes; that certainly a different kind of teaching process is required to produce the creative mathematician or the kind of person who can use mathematics to solve problems that have not been solved as yet or who can use mathematics to solve problems better than they have been solved. This goal requires something more than what we ordinarily think of as "teaching mathematics". I am not going to say what that teaching is or should be; I am simply going to make these distinctions as background that I will come back to by way of conclusion.

We, in our research on teaching, have started out with what is probably the simplest aspect of teaching. We have tried to find something called the technical skills of teaching. By this we mean those kinds of teaching performances which are repeatable and that you can train somebody to use in a wide variety of circumstances. At present we have not developed skills which are specific to the separate subjects of the curriculum; rather we have tried to pick skills which,

by and large, most teachers could use at some time in the course of their teaching.

For example, we assumed that some kind of a dialogue between teachers and students is an integral and desirable aspect of almost all teaching. We then asked the question, "How can the teacher stimulate this dialogue?" It seems to me there are two things the teacher can obviously do. The teacher can use certain kinds of leading questions to elicit pupil behavior and the teacher can reinforce that behavior when it occurs. So one of the technical skills we teach beginning teachers is the skill of reinforcing a student whenever a student participates in class. We try to bring the teacher to the point where he or she is very sensitive to the fact that the student is engaging in dialogue and will reward the student for the act of engaging.

Some of the other things we attempt to teach are to differentiate between kinds of questions a teacher might use to elicit pupil behavior. We are developing a skill called "higher order questioning". If you ask me to describe it, I usually do so by saying that it is learning the difference between "Who killed Cock Robin" and "Why was he killed?" We try to teach teachers to make a distinction between these two kinds of questions, the higher order question being the "why" type of question involving reasoning rather than repeating factual information.

We have developed a small list of these skills. We develop the idea of a skill in part by intuition about what we think is required and in part by thinking about what constitutes good instruction. We do it by trying it out and determining whether the performance we think is teachable, in fact, turns out to be teachable.

Let me illustrate how we might go about developing such a skill. In the teaching of mathematics, for example, it has been customary to advocate heuristic methods. There is a literature on the topic and quite a bit of investigation. In this literature, the kinds of behavioral processes or the kind of teacher behavior involved in heuristic methods are spelled out in rather general terms. One of the most difficult things to do when studying heuristic methods is to describe the teacher behavior so that you know what a teacher is supposed to do concretely when he is using a heuristic teaching procedure.

Now, if we were developing a technical skill involving heuristics, we would spend considerable time trying to describe in great detail how a teacher would act when he or she is using a heuristic method. We would also watch a teacher using such methods over a long period of time. In the attempt to tell a teacher how to use heuristic methods we would learn what we mean in behavioral terms by heuristic methods.

This may seem like a "back-door" way of doing things, or it may appear "low order". I have the feeling that somehow there ought to be a theory for this analysis, but there is none sufficiently specific for this task. However, we have learned that the kind of technology we are developing for training is also a very useful way of making much more precise the kinds of teaching behavior covered by labels such as "heuristic teaching" or "discovery learning".

We have been proceeding in this way—defining what we call technical skills and training teachers to use them. The outcome, hopefully, will be a basic set of skills that can be communicated to teachers. The extension of this work to the teaching of mathematics is obvious. The next step is to define and describe those skills which are specifically and more generally required in mathematics than they are in some other subject, then use our training methods to teach teachers how to use these particular instructional methods. I think that at the present time, enough is known about the teaching of mathematics that it would be easy to move from where we are today to a specific description of the kinds of skills that are specifically required in teaching mathematics.

I would hasten to point out that such skills probably would not include the whole sum of activities involved in teaching mathematics. But they would be the basic processes by which a teacher is most likely to generate the kind of pupil behavior now thought necessary to acquire mathematical knowledge.

The other part of our work has been on the training methods. Actually, most of our experimentation has centered on this. What I have been talking about up to this point are the dependent variables in the experiments we do. We are primarily interested in how to produce these skills in teachers.

We have devised a technology for experimenting which I would like to describe briefly. One of our prime technical devices is the video recorder. We tape teacher performances with a portable video recorder. This video provides us with a permanent record of a teacher's behavior. It also provides us with a record to analyze that behavior and, as I will describe shortly, it provides us with a way of talking to teachers about their teaching behavior.

The second approach to training that we have taken is to break down the teaching act into small units. Successive acts of teaching in these small units we call "micro-teaching". We present a teacher with a teaching situation in which he or she teaches a small number of students for a relatively short period of time. For some obscure reason we think in terms of the number "5". We use five students and five minutes of teaching. However, this format is almost infinitely adaptable. You can reduce the number of students to one; you can increase it to ten or fifteen. You can instruct any kind of a student group. You can also use longer periods of time.

The advantage of using the brief period with a small number of students is that this is a good way to initiate somebody into a training program, as we reduce the threat of the regular classroom and all the kinds of things a teacher has to keep under control simply to manage an ordinary classroom. Secondly, we ask the teacher to do relatively few things and, as a consequence, she does not have to prepare for several hours in order to teach a one-hour period. She can prepare in about a half-hour to teach for five minutes. By using these shorter teaching sessions, we get more training per unit of training time. We also train under conditions which are less threatening particularly for beginning teachers.

We apply this technology to teaching technical skills. We begin by filming the performance of a teacher, then we do something in the way of teaching—either by sitting down and talking to the reacher or by showing him the teaching performance of a model teacher. The trainee practices again for a short period of time; we talk to him again. The trainee practices a third time; after several training trials we terminate the sequence. At the end of the sequence, the trainee has learned the elements of the skill fairly well. The next step is to use

the skill in more complicated circumstances and to combine it with other kinds of skills.

With this kind of training you can produce the kind of teaching behavior you think is necessary to produce certain kinds of pupil effects. What we have been experimenting with has been the training variables, that is, those training elements which will produce the learning of the skill. We have found, for example, that it is far more effective to sit with a teacher and reinforce the teacher for doing certain things as he watches his video taped performance than it is simply to have the teacher watch his or her own performance without assistance. You get far more change if you supply reward and some advice on how to improve than you do in any other way that involves the teacher watching his performance.

We have also developed a series of model films for this training. We are interested in studying the best combination of model characteristics. We have used positive models; that is, teachers who are using a skill in the desired manner. We have also used negative models. We are planning to experiment using one's own best performance as a contradictory kind of model. We experimented with the prestige characteristics of the model, by saying. "This is an experienced teacher", or something else of that kind to stimulate interest in imitating the behavior of that teacher.

We will do an experiment during the winter quarter in which we will try something different. Instead of showing just the teacher behavior, we are going to show the pupil behavior and see whether a teacher learns the desired teaching behavior better by watching another teacher or by watching students do what he hopes they will do. This is a study in which the teaching behavior is "inquiry" training and the analog in mathematics is "discovery" learning. The same experiment could be done using mathematics as the subject matter.

We have performed about seven or eight experiments and, by and large, we have demonstrated that some forms of presenting models are more effective than others and that some feedback conditions are more effective than others. We hope to find the

maximum combination of these variables in relation to a particular kind of teaching skill.

By way of conclusion, what I want to point out what I hope is obvious by now, that the technology and also the theoretical knowledge we have developed is readily adaptable to the training of mathematics teachers. This development requires some selection of those kinds of teaching skills that mathematicians and educators feel are necessary skills for producing certain kinds of pupil behavior.

The consequence of this will be twofold, if and when it is done. One is that we will learn what must be learned in order to teach mathematics effectively. We will also learn the connection between teaching behavior and the desired pupil behavior. In any of these experiments it is possible to do an intensive analysis of what the pupil has learned. So many of our assumptions about what a teacher ought to do in order to produce a certain kind of pupil behavior can be checked quite directly by studying whether that effect is produced.

If you will relate these suggestions to my original distinctions, it seems to me the goal is to decide which among these many objectives—the teaching of formal mathematics or teaching to think quantitatively or producing the creative mathematician—has priority, and, also, to try to define (or describe) the instructional methods that ought to produce these changes. Once that process is carried along sufficiently, it seems to me that we are at the point where it can be refined by using the kinds of technology and experimentation I have been describing.

Dr. John E. Coulson, System Development Corporation, Santa Monica, California

Perhaps I am here under false pretenses, because I cannot tell you very much about how students learn mathematics. I am not even certain that it is entirely meaningful to ask how students learn mathematics as distinguished from other content area. One reason is that if you look in fine detail at the learning process of students studying mathematics, you find a tremendous diversity of activities within the field, and even within a half-hour lesson in one subdivision of mathematics. The activities required are so different from one

course to another, even from one minute to another, that it may never be meaningful to talk about how people learn mathematics in general or how they can best be taught mathematics. What we *can* do is to apply empirical research procedures to develop new materials that are demonstrably effective in teaching specific mathematical skills such as adding two inequality statements. This is an approach we have taken in much of our work at System Development Corporation.

A second reason for the difficulty in talking about how students learn mathematics, *per se*, is that the actual skills involved in mathematics cut across many other subject areas as well. The same kinds of learning that occur in mathematics are also found in social studies, science, reading, or almost any other area you care to mention. To illustrate this point I would like to describe a study we did about two years ago, working with programmed materials in the four different content areas. One of the areas was introductory reading, another was set notation for first graders, a third was Spanish at the junior high level, and a fourth was high school geometry. We were looking for general principles or rules that would tell us what factors in the design and sequence of programs make a difference in student learning. We were looking for generalizations that cut across all the subject areas, but we were also looking for systematic differences.

We wanted to know whether there are consistent differences in the way you should structure the lessons for young, elementary school children as compared with older students at the junior high or high school level. Similarly, we tried to find systematic differences between the reading and Spanish materials, which we labeled "verbal", and the arithmetic and the geometry materials which we labeled "mathematical/quantitative". Although we came out with some fairly consistent differences as a function of age group, we found no systematic differences between the "verbal" and "quantitative" material. The reason for this is that the arithmetic and geometry work depends to a tremendous extent on the same kinds of verbal skills that you find in reading and Spanish. Because so many of the critical skills cut across different subject areas, I believe that the best theoretical guidance that psychologists or other researchers will be able to give mathematics teachers in the future will pertain to these basic, content-independent skills.

Let me give you an example of one of the content-independent skills I am talking about. You can teach a child to discriminate a triangle from a circle. You show him the figure of a triangle and he will distinguish that from another shape labeled "circle". hen you can teach him that a whole variety of 3-sided figures that look different in many respects are all classified as triangles. Having taught him to generalize the concept of triangle, you can then teach him to discriminate between an isosceles triangle and a scalene triangle or an equilateral triangle. These same key processes of generalization and discrimination are found in other areas as well. For example, a child may learn to distinguish between animals and plants, then to generalize the concept of animals which includes a large variety of organisms, and finally to discriminate between, say, insects and fish. I would expect the same kinds of procedures to be good in teaching these generalizations and discriminations in one content area as in the other and I would expect to look at the same kinds of learning processes.

Similar learning activities will be found in the verbal area, e.g., teaching verbs versus nouns, teaching that a whole collection of words or of word combinations can be verbs, and then teaching irregular versus regular verbs. And we know a few things about teaching generalization and discrimination. We know, for instance, that you are wasting your time if you just give a long series of positive examples without any negative examples of the concept being taught. If you start the child with many pictures of different triangles, but do not introduce any figures that are not triangles, you will lose time in teaching the concept of triangles. Similarly, we know that you usually get more effective learning if you start with a fairly gross kind of discrimination (circle vs. triangle) and then work toward progressively finer discriminations (isosceles vs. scalene triangle). These principles are probably not new to you but they do illustrate the level at which I believe psychologists and learning theorists will be able to give guidance about the learning and teaching of mathematics. We need to look much more closely at the component skills involved in solving mathematical problems. Is the child learning to analyze, to break up a problem down into its component parts? Is he learning to synthesize, to put together a number of more basic

skills into some new skill? Is he learning to apply some specific facts to a particular problem? These are categories of learning that we may be able to come to grips with. Perhaps eventually we can come up with some sort of 3-dimensional matrix, which on one dimension will show different content matter, on another dimension will show the kind of skill involved, and on the third dimension will show teaching techniques or even specific teaching materials. And perhaps a curriculum might be constructed by selecting and sequencing cells out of this 3-dimensional matrix. This may represent an idealized approach, though certainly we are a long way from being able to produce that type of matrix now.

My other general observation of learning in the content area of mathematics, though this certainly applies equally to any other content area, is that students do not make the large conceptual leaps that many people would like to feel they do. It is probably misleading to believe that a student can take two or three elements that he has learned, and somehow combine those elements for the first time with some new element that he has not been taught specifically, and put this all together to come up with a huge "insightful" leap forward. It is a convenient belief because it takes much of the burden off the teacher. The teacher does not have to worry about whether he has carefully sequenced the learning experiences to lead the student from one sub-goal or sub-objective to the next until the student achieves the more complex skills. The teacher having this optimistic faith in large conceptual leaps can assume that he need only lay out a few basic ingredients and the students will somehow add a new ingredient and come out with some higher level concept or solution. When such leaps appear to happen, usually one of two things has happened. Either the students have not really learned as much as the teacher thinks they have, or they have already had practice in similar tasks outside that teacher's classroom so that the apparent leap is only illusory.

One example may help to give more substance to my claim that students do not learn as much as most teachers believe they do. We developed tests to determine how much students were learning from three different courses currently being used to teach foreign languages in many junior high schools throughout California. These were not the

usual normative tests designed only to create a performance spread, but were criterion-referenced tests constructed to assess performance on every specific objective that the course authors claimed the courses were intended to teach. We found that performance at the completion of the courses averaged around 50–60 per cent of the specified objectives (i.e., about half the test items were answered correctly). I am sure that this finding was a shock to most of the teachers in those schools. Although this particular example was in the language area, my observations lead me to feel that the results would not be substantially improved if the same study were conducted in arithmetic or algebra.

In mathematics or in any other subject area, I know of only one way to ensure that instructional materials will teach what they are designed to teach. That is to define precise course objectives, then to develop good diagnostic instruments to test performance on every one of those objectives, and finally to apply trial-and-revision procedures to the materials as often as needed to bring those materials to the specified performance standards. I feel that much of the difficulty with a lot of the new curriculum materials is that all of the effort has been put into working out the overall curriculum design and very little effort has gone into the necessary, but tedious, trials and revisions of the materials. The result is that the materials have gaps that the students are unable to fill by themselves. To be sure, some students may supply the missing steps on the basis of information gained outside the classroom, but others will experience failure at critical points in the course. Neither the teacher nor the designer of instructional materials should depend on "accidental" learning; rather, he should attempt to present a carefully planned, empirically validated sequence of learning experiences that will allow every student to master the essential course objectives.

Professor Zoltan P. Dienes, Université de Sherbrooke

I must disagree with practically everything which has been said up to now. The main reason for this comes no doubt from the use o terms. We do not understand the same things by the same words. Probably when Mr. Coulson or most of you talk about mathematics learning, you understand a verbal kind of skill whereby the carrying

out of certain operations under certain conditions is learned as well as the solution of certain kinds of problems. Mathematics learning is not necessarily of this kind, although the "other kind" is still pretty rare in the world. There are only a few islands where it can be seen.

So we cannot blame the textbook writers or the psychologists for thinking what they are thinking. Psychologists can only "explain" what they observe, and it is small wonder that a rat-type learning model is so popular, since we seem to teach children much as we teach rats.

This verbal, associative version of mathematics learning is so prevalent that from a truly mathematical point of view, it appears that, in effect, very few children learn mathematics at all! What they learn are certain stimulus-response-outcome triads. It is quite obviously right that we should have contrast in our teaching, as we have heard, but what we forget is that mathematics is much more complex than any kind of other activity with which the child is called upon to get familiar with in school. For example, the fact that there are certain inclusion properties which he has to face makes contrast necessary, and so exclusion properties must also be learned. With this we should all agree, at least in theory, and no doubt many teachers put this principle into practice. But mathematical structures are enormously complex and much more care needs to be taken with them than with the learning of simpler structures. What actually happens is that children are simply taught how to carry on certain procedures. There is very little difference between the teaching of the so-called new math, old math, or any other kind of math children have learned in schools. They just make different noises now from what they were making before and they are taught to do different kinds of problems than what they were taught before. The actual structures do not appear to be getting at all clearer for the children, in the sense that a mathematician understands them. After this preamble, I hope I might be permitted to say a few words about what kind of research *can be done* on structural learning, which is non-trivial and which is such that we can have a behavioral criterion according to which we can say that a person has or has not mastered a certain structure.

I cannot go into detail because of lack of time, but clearly the kind of task which we have to give subjects should be suitable for children and adults. The task must be unfamiliar to them, in order to conduct experiments, and it should also be suitable to a wide range of children as well as of adults. It should not be too difficult for children; it should not be too easy for adults. It should be challenging for the very bright and it should also be possible for the less bright. It should also be suitable for standardizing so that we can make some comparisons, and yet it should also allow sufficient freedom for different strategies to be applied.

I have no time to explain how all these kinds of problems are solved in the technicalities of the laboratory but I hope to talk about that more at another time. Some very obvious and easy structures in mathematics which satisfy the above conditions are mathematical groups or rings of between two and twelve elements each. We have found in laboratory research that it is, indeed, quite possible to set up this kind of learning, even through a machine where what the subject does is press a few buttons. In this case he is "rewarded" by a correct answer, and "punished" by an incorrect one.

Essentially what happens is that we begin with a machine in a certain *state*; the subject operates a certain *operator* and then must predict the next state. This prediction will be either right or wrong depending on the program that has been programmed into the machine. Now, the next state is the state of the machine and the subject has to play the game of operating against that state and predicting the next state. When he makes a certain number of correct predictions in a row, the machine automatically examines him. Either he reaches a certain criterion in the test or he does not. If he does, the bell rings and he is told to go away. If he does not, the buzzer rings and he can play against the machine again until he reaches the criterion for the machine to examine him; and so it goes on.

In this procedure it is possible to leave sufficient room for *observation of strategies*. We found, for example, that there are quite distinct strategies employed by subjects. There are developmental differences and there are even sex differences in the kinds of strategies that are applied. To obtain a check on the "reality" of these strate-

gies, scores were developed which measured exactly the extent to which subjects used this or that strategy. We also used the subjects' own evaluations of what they considered the game to have been, pitting these against the strategies which we measured. An example of the strategies is the following; the *operational strategy*: such a strategy was evidenced by the subject trying the same operator several times over to see what happened to the state. Such subjects on the whole gave an *operational evaluation* to the game. The subjects that played another way on the whole gave corresponding evaluations for the games. And so one interesting concrete result was that it was possible to say that certain types of subjects tended to use certain types of strategies and that they were to some extent conscious of this. I think this obviously has some bearing on the educational situation. In the normal schoolroom no strategies are allowed—what the textbook or what the teacher says must simply be regurgitated.

There were some other interesting details that we found. For example, we used the order of presentation of the task as one of the independent variables: simple, first—complex, afterwards; and complex, first—and simple, afterwards. So far the complex to simple has won hands down all the time, except for the adults. If children have to learn a 4-element group (whether it is the Klein-four or the cyclic-four group), they find it easier to learn the 4-element group first without first having to learn the 2-element group. If they have learned the 2-element group, they then have to generalize and this is difficult for them. On the other hand, if they have already learned the 4-element group first, they have *with it* learned the 2-element group and so have learned to recognize it when it turns up. In other words, there is evidence that there is negative transfer from simple to complex, and there is positive transfer from complex to simple. This is one point where I must disagree strongly with Dr. Coulson.

Our results tend to show experimentally that children *do make leaps*. Encouraged by this result, I tried some applications of this principle in the classroom. For example, I found that children got on very well with learning the complex algebra before the real algebra, and the end result was much more satisfactory because they learned what they learned in context. Another thought led us to wonder

why we started with one-dimensional vector spaces when we can start with multi-dimensional ones. This also works out very well with children as young as nine years old. This is just one application from the classroom, as well as from the laboratory, that certain radically different procedures, as against what you have been hearing, are indicated if we are going to make any kind of headway in making our mathematics teaching more efficient.

There are a few other pieces of evidence. One of these relates to the handling of symmetry by children. By symmetry I do not only mean geometrical symmetry, I mean structural symmetry in general. Supposing we have three elements a, b, and c which generate one another; if ab together generate c, and bc generate a, and ac generate b, then we have complete symmetry of these three. But if only two of these things take place, without the third, then we have some kind of asymmetry about the situation. We have found that learning the Klein group is easier than the cyclic group for this reason. Clearly the handling of symmetrical situations, ways of departure from symmetry, and allied problems need to be investigated as a part of any plan of attack on the problems of mathematics learning.

We went on to more sophisticated experiments along this line in the last year or so. In these experiments, we saw each subject four times on four different days and gave them different structures to learn. This allowed us to ring the changes even more and in more exciting ways than before and to study such questions as the effects of the inclusion of a structure in another, of overlapping, of identity, of generalization, etc. We tried to explain the students' behavior in learning these structures in terms of stimulus-response-outcome models. For example, we found that the five group followed by the three group was invariably easier for children than the three group followed by the five group. And this is not an inclusion problem, as in the two-four and the four-two (you see, the two group is included in the four group but the three group is not actually included in the five group, nor is it isomorphic to any part of it). So this was a different problem and yet it still came out the same way. On the stimulus-response-outcome prediction there is no reason to expect any difference, as the "same amount of learning" has to be done. By "amount"

here we mean the stimulus-response-outcome triads. This is, in fact, how the adults performed. It seemed that the adults had already been sufficiently brainwashed by the process known as education so that they performed as this mechanism predicted. "Education" consists mostly of learning S-R bonds and hence such a result is only to be expected.

On the other hand, we can use a model which is not a stimulus-response-outcome model, but a *role model*; we might suppose that a certain operator within a structure has a certain role and the subject learns what that role is. If an operator with such a role occurs in another structure, then this role is transferred. If we use different symbols in the two structures, then different stimulus-response-outcome triads must be learned in the second structure and, according to the S-R-O theory, this is new learning. According to the role model, it is not new learning. The children's performance can more parsimoniously be "explained" by a role model.

Various other relationships such as embeddedness are very interesting to investigate. For instance, a structure can be embedded in another structure not only simply but multiply (it can have several isomorphic images in it). We found again that children coped with this multiple embeddedness relatively very much better than adults did, as opposed to generalization, which the children were not so able to do. So one of the results of our experiments is, if I may conclude, that generalization, that is, going from the simple to a somewhat more complex structure, is very much harder for children. And yet this is what we do with them most of the time! This is no doubt because we think that they are like ourselves and that they will learn a little, followed by a little more and so on. Each time we are putting them into a situation of generalization which for them is extremely difficult. If we would only allow them their *leaps*, they might well learn more effectively.

Dr. Becker

Thank you, Professor Dienes. We have reached the end of our time limit, unfortunately. Actually, I feel that this is where we should begin. We have heard here today some good points and observations made that bear on the teaching and learning of mathematics. It

would be valuable to discuss many of these in more detail, but time does not permit.

On behalf of the California Mathematics Council, Northern Section, and Phi Delta Kappa, I would like to thank each of the panel members for their participation. We will conclude on the note that meetings such as this are good for us as mathematics educators.

Thank you for coming, we hope it has been valuable.

Reaction to Professor Dienes by Dr. Coulson

Because of time constraints neither Professor Dienes nor I have had a chance to develop his thesis very thoroughly. Without a more detailed description of the learning situations that Professor Dienes has alluded to, I cannot tell whether we actually have a disagreement about definitions, about empirical data, or about the interpretation to be given the data. To have more than an abstract, philosophical discussion, we would need to examine every step in the instructional sequence to see just what cues were contained in Professor Dienes' directions to the students, what response alternatives were provided, what contingencies governed the sequence of problems presented, etc. In fact, for thorough understanding of the learning conditions we would need to know what other problems, requiring similar responses, the students might have encountered prior to their exposure to Professor Dienes' problems.

I suspect that our difference in approach is partly temperamental. I look immediately for a trail of learning experiences, both within and outside the experiment itself, that could reasonably have served the student as a bridge from one large step to the next. Professor Dienes, perhaps, prefers to emphasize the large steps themselves and to pay less attention to other learning experiences that he may regard as incidental, or outside the scope of the experiment.

In any event, I would expect that a full analysis of the learning situation involved in Professor Dienes' studies would show that his students' apparent ability to make conceptual leaps depended, in actuality, upon other learning experiences, many outside the school, that helped the students transform each large step into a number of

much smaller steps. The fact that a response may occur very quickly and "spontaneously" does not argue against my premise, because there is a long history of experimental evidence that "spontaneous" responses may be long chains of simpler responses.

Reaction to Professor Dienes by Professor McDonald

Dr. Dienes is obviously discussing the basic learning problem usually called by psychologists "transfer of learning" or "transfer of training". The main thrust of his argument seems to be that his data show that training on complex tasks facilitates learning of easier tasks, particularly for children. He seems to think that his evidence is data for some other theoretical model of learning than a stimulus-response model. I would argue that the data are not any more consistent, in the form in which they are presented here, with any other paradigm for learning than they are for a stimulus-response model.

It is simply not true that the stimulus-response model predicts that transfer moves in the direction from simple tasks to complex tasks. There is experimental literature showing that the reverse may be true. Positive transfer from more difficult tasks to simpler tasks usually occurs when the more difficult task includes the components or elements of the simpler task so that the learner is practicing on these elements as he works on the more difficult task. Without further information, which time did not permit Dr. Dienes to provide, it is difficult to tell whether or not his tasks are sufficiently different that an argument can be made that the data are consistent with some other paradigm or whether the tasks are, in fact, inclusive. His example of teaching children complex algebra before they learn "real" algebra suggests that the children are learning the elements of "real" algebra while learning the complex algebra.

Running through Dr. Dienes' presentation are references to such concepts as "structure" and "embeddedness". It is difficult to tell what kind of learning may be occurring without fairly precise definitions of these concepts. Admittedly, it is startling that on these tasks children learn more easily than adults, but this fact cannot be properly evaluated without knowing, as was said above, something about the nature of the task and the relevance of the task to such

concepts as *embeddedness* and *structure*. The issue is, not what interpretations are to be made of these data, but what is happening. It appears that greater positive transfer occurs for children in moving from complex to simpler tasks. However, the difference between children and adults may be that adults have practiced the relevant tasks in a variety of ways, or have responses relevant to solving task problems, so that transfer effects are attenuated. Stated otherwise, there may be more for the children to learn, therefore, more evidence of positive transfer. It strikes me as unfortunate that these facts are treated as an argument against a transfer position which might as readily account for them, or more readily account for them than some other kind of learning paradigm or a developmental paradigm.

Note by Dr. Dienes

Since the structures learned by the children in the Adelaide experiments described earlier are group structures, including groups of 6 and of 9 elements, it seems entirely improbable that any prior learning would have taken place which with any stretch of imagination could be said to have helped in learning such abstract structures. In fact the reason for the choice of such structures was just so as we can avoid this very type of objection.

Quite clearly in the complex to simple treatment of any *type of task*, there are bound to be some elements of the simple task included in the complex task. Nobody in his senses would claim that analysing Hamlet would cause positive transfer effects on learning the multiplication tables. It is clearly almost unheard of in educational practice to proceed from the complex situation to the simple, and naturally much research needs to be done as to the optimum degree of complexity at which a certain type of task can best be introduced.

The detailed descriptions of the tasks, together with the definitions of the terms, called into question by Professor McDonald, appear in "The effects of structural relationships on transfer", by Z. P. Dienes and M. A. Jeeves, published by Hutchinson Educational, London, England.

About the participants

Professor McDonald received his Ph.D. in Education and Psy-

chology from Stanford University. He has been active in research in teacher behavior, has written many articles, and is author of a book, *Educational Psychology*. Presently he conducts research in the Stanford Center for Research and Development in Teaching.

Dr. Coulson received his Ph.D. in psychology at Columbia University in 1956. He joined the RAND Corporation in 1956 and System Development Corporation in 1958. For eight years he has conducted research and development in programmed learning, computer-aided instruction, and the design of instructional management systems. He has authored a large number of articles on programmed instruction and edited a book on the subject.

Professor Dienes received his Ph.D. in mathematics from London University in 1939. Since then he has studied psychology and has been involved in a number of projects in various parts of the world. He has authored many articles, several books, and has developed children's learning materials in mathematics.

Dr. Becker received his Ph.D. in mathematics education from Stanford University in 1967. Before coming to Rutgers—The State University he was on the staff of the School Mathematics Study Group (SMSG) for three years, where he was involved in the work on the National Longitudinal Study of Mathematical Abilities (NLSMA).

Lloyd V. Rogers is an elementary school teacher with interests in mathematics learning and related research. He has been active in organizations of mathematics teachers and has experimented with many new mathematics materials for elementary school children.